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Graded contractions of the Lie algebra $e(2,1)$

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Abstract. The graded contractions of pseudo-Euclidean Lie algebra $e(2,1)$ are studied. The non-equivalent gradings of $o(2,1)$ of type $\mathbb{Z}_3$ and a $\mathbb{Z}_2\times\mathbb{Z}_2$ are extended to the entire Lie algebra $e(2,1)$, using the action of $o(2,1)$ on the Abelian ideal (the translations.). The graded contractions embed $e(2,1)$ into a large family of six-dimensional Lie algebras. The family includes solvable, nilpotent and nonsolvable Lie algebras, both decomposable and indecomposable ones. The distinction between graded and Inönü–Wigner contractions is analysed. The physically most interesting Lie algebras obtained by the contractions are the inhomogeneous Galilei and pseudo-Galilei.

1. Introduction

Lie algebra contractions were introduced into physics by Inönü and Wigner [14] as a mathematical expression of a philosophical idea, namely the 'correspondence principle'. This principle tells us that whenever a new physical theory supplants an old one, there should exist a well defined limit in which the results of the old theory are recovered. More specifically Inönü and Wigner established a relation between the Lorentz group and the Galilei group in which the former goes over into the latter as the speed of light satisfies $c \to \infty$. As a mathematical concept, contractions were already used somewhat earlier by Segal [28].

The theory of Lie algebra contractions (and deformations) has acquired a life of its own. It provides a framework in which large sets of Lie algebras can be embedded into families depending on parameters. All algebras in such a family have the same dimension, but they are not mutually isomorphic [18]. The embedding also provides relations between the representation theories of different Lie algebras in the same family.

Very often the contraction procedure starts from a grading of a simple Lie algebra L [25, 10], the representation theory of which is well known. The contraction then leads to solvable Lie algebras, or Lie algebras that have a non-trivial Levi decomposition, i.e. a non-trivial maximal solvable ideal (the radical) [17]. Typically the aim of the contraction procedure is either to preserve a chosen subalgebra A while contracting everything else in L in every possible way [19, 20], or to classify the outcome of all possible graded contractions of a given Lie algebra [1, 8], as in this paper. Note that the process advances in the direction of the Abelian

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algebra. That is, some of the subspaces, which were not commuting before a contraction, will commute after it. (It would be possible to study the process in the opposite direction starting, say, from an Abelian algebra, but that would be an entirely different problem calling for different strategies.) During successive contractions corresponding to refinements of gradings, one may ask whether the final outcome may differ if obtained in one step ($A \to C$) or in two steps ($A \to B \to C$). The answer depends on the case. It is easily found from the graph of successive refinements of the gradings in each specific case: imagine the graph oriented in the direction of refinements. Then if there is a link, following the directions, between B and all the outcomes of $A \to C$, there is no difference; otherwise there is a difference between the two ways of contracting A.

Contractions have been used to develop the representation theory of such Lie algebras, in particular calculate Clebsch–Gordan coefficients and transformation matrices for non-semisimple Lie algebras [13].

The close relationship between special function theory and group representation theory is well known [30]. Recently the theory of Lie algebra contractions has been used to relate the special functions obtained when separating variables in Laplace–Beltrami operators on spheres, or hyperboloids, to those obtained in Euclidean, or pseudo-Euclidian spaces [15, 16].

Among other recent applications of Lie algebra contractions, we mention the shell structure of deformed atomic nuclei [6].

The concept of contraction has recently been extended to the theory of quantum groups, where it has proved to be very fruitful [7, 4].

The original Inönü–Wigner contractions can be interpreted as singular changes of basis in a given Lie algebra L. Indeed, consider a basis $\{e_1, \ldots, e_n\}$ of L and a transformation $f_i = U_{ik}(\varepsilon)e_k$, where the matrix U realizing the transformation depends on some parameters ε. For $\varepsilon \to 0$ (i.e. some, or all of the components of ε vanishing) the matrix $U(\varepsilon)$ is singular. In this limit the commutation relations of L change (continuously) into those of a different, non-isomorphic, Lie algebra L' [14, 27, 9, 32].

A different approach has been developed more recently, namely that of graded contractions [21, 23, 8, 24, 1]. The Lie algebra L is first decomposed into eigenspaces of an automorphism of L:

$$L = L_0 \dot{+} L_1 \dot{+} \cdots \dot{+} L_{N-1} \tag{1.1}$$

$$[L_i, L_j] \subseteq L_{i+j}. \tag{1.2}$$

The commutation relations of L are then modified in a manner respecting the grading

$$[L_i, L_j] \to [L_i, L_j]_\varepsilon = \varepsilon_{ij}[L_i, L_j] \tag{1.3}$$

where ε_{ij} are some constants, subject to the condition that $[L_i, L_j]_\varepsilon$ be a Lie bracket (see below).

Since gradings can be introduced in a systematic and exhaustive manner, this approach makes it possible to study contractions in an equally systematic way. Until now, graded contractions have been considered mainly for simple Lie algebras L.

The purpose of this article is to apply the concept of graded contractions to an affine Lie algebra, that is the semidirect sum of a simple Lie algebra and an Abelian one, on which the semisimple subalgebra is represented faithfully and irreducibly. The considered Lie algebra is the pseudoeuclidian Lie algebra

$$e(2, 1) \sim o(2, 1) \niplus T(3) \tag{1.4}$$

and we analyse one grading of this algebra in detail, namely a $\mathbb{Z}_3$ toroidal grading. This is a grading induced by an order three finite subgroup of the maximal torus of $o(2, 1)$.

The study of contractions of the Lie algebra $e(2,1)$ can serve as a model for the analysis of other non-semisimple Lie algebras, in particular the general pseudo-Euclidean Lie algebra $e(p,q)$. In principle, contractions of $e(2,1)$ could be studied as part of an analysis of contractions of various real forms of the algebra $o(N,\mathbb{C})$, $N = p+q+1$. The present study, as well as previous ones for $sl(3,\mathbb{C})$ [8, 1] show that graded contractions of a given Lie algebra L lead to a large variety of different Lie algebras. In order to make the systematic investigation of graded contractions manageable for arbitrary dimensions it is necessary to make use of successive refinements of gradings [1, 33]. In particular, $\mathbb{Z}_2$ contractions of $o(p,1)$ can lead e.g. to the Euclidean algebra $e(p)$, or pseudo-Euclidean algebra $e(p-1,1)$. Thus, e.g. graded contractions of $e(p-1,1)$ are refinements of $\mathbb{Z}_2$ contractions of $o(p,1)$ [22].

More specifically, the contractions of $e(2,1)$ in this article form a specific class of graded contractions of the Lie algebra $o(3,1)$ of the homogeneous Lorentz group $O(3,1)$.

For somewhat related recent work involving graded contractions and inhomogeneous classical Lie algebras we refer to de Azcarraga *et al* [3, 2].

Section 2 is devoted to the gradings of the simple Lie algebra $o(2,1)$ and we introduce three inequivalent gradings. These gradings are extended to the entire Lie algebra $e(2,1)$ in section 3, yielding five mutually inequivalent gradings of $e(2,1)$. The problem of graded contractions of affine Lie algebras is formulated in section 4. The main results of this article are summed up in sections 5 and 6. One of the gradings of $e(2,1)$, namely the $\mathbb{Z}_3$ toroidal grading is used and all contractions preserving this grading are constructed in section 5. A total of 23 mutually non-isomorphic Lie algebras is obtained and they are identified using basis-independent criteria [26]. Section 6 is devoted to contractions corresponding to a fine non-toroidal grading. Some conclusions are drawn in section 7.

For completeness and uniformity of exposition we present and analyse all outcomes of graded contractions of $e(2,1)$. Hence we include some results that were previously known [22, 32, 11, 12, 29], especially in sections 6.1–6.3.

2. The Lie algebra $o(2,1)$ and its gradings

The subalgebra $o(2,1)$ of $e(2,1)$ is the Lie algebra of the homogeneous transformations which preserve the Lorentzian metric $(1,-1,-1)$. In this article we choose the invariant form as follows:

$$(x,y) = x^T K y = (\,x_1 \quad x_0 \quad x_{-1}\,)\begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ y_0 \\ y_{-1} \end{pmatrix} = x_1 y_{-1} + x_{-1} y_1 - x_0 y_0. \tag{2.1}$$

Note that the eigenvalues of K are indeed $+1, -1, -1$.

The natural three-dimensional representation of $o(2,1)$ is defined using the matrix K chosen in (2.1),

$$o(2,1) = \{X \in \mathbb{R}^{3\times 3} \mid XK + KX^T = 0\}. \tag{2.2}$$

Thus a generic element of $o(2,1)$ is of the form

$$\begin{pmatrix} b_0 & b_1 & 0 \\ b_{-1} & 0 & b_1 \\ 0 & b_{-1} & -b_0 \end{pmatrix} \qquad b_1, b_0, b_{-1} \in \mathbb{R}. \tag{2.3}$$

Putting one of the parameters equal to one and the others equal to zero, we have a basis of $o(2,1)$ as

$$B_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad B_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad B_{-1} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \tag{2.4}$$

The commutation relations and the Casimir operator are, then, respectively

$$[B_0, B_1] = B_1 \qquad [B_0, B_{-1}] = -B_{-1} \qquad [B_1, B_{-1}] = B_0 \tag{2.5}$$

$$C = B_1 B_{-1} + B_{-1} B_1 + B_0^2. \tag{2.6}$$

The first $\mathbb{Z}_2$-grading of $o(2, 1)$ is the decomposition of the algebra into eigenspaces of the transformation

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} b_0 & b_1 & 0 \\ b_{-1} & 0 & b_1 \\ 0 & b_{-1} & -b_0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} b_0 & -b_1 & 0 \\ -b_{-1} & 0 & -b_1 \\ 0 & -b_{-1} & -b_0 \end{pmatrix}.$$

Clearly the eigenvalues are ± 1. Thus the eigenspaces L_0 and L_1 corresponding to eigenvalues $+1$ and -1 respectively are generated by

$$L_0 = \{B_0\} \qquad L_1 = \{B_1, B_{-1}\}. \tag{2.7}$$

The second $\mathbb{Z}_2$-grading of $o(2, 1)$ is the decomposition of the algebra into the eigenspaces of the transformation

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} b_0 & b_+ + b_- & 0 \\ b_+ - b_- & 0 & b_+ + b_- \\ 0 & b_+ - b_- & -b_0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -b_0 & -b_+ + b_- & 0 \\ -b_+ - b_- & 0 & -b_+ + b_- \\ 0 & -b_+ - b_- & b_0 \end{pmatrix}.$$

Using again the notations of (2.4), we have

$$L_0 = \{B_1 - B_{-1}\} \qquad L_1 = \{B_0, B_1 + B_{-1}\}. \tag{2.8}$$

The gradings (2.7), (2.8) are non-equivalent, for example, because the subspace L_0 in (2.7) corresponds to a non-compact group, while it corresponds to a compact group (2.8).

The $\mathbb{Z}_2$-grading (2.7) can be refined into a $\mathbb{Z}_3$ one by splitting the two-dimensional subspace into two. For that we use an automorphism of order 3. Denoting $\omega = e^{2\pi i/3}$, we have

$$\begin{pmatrix} \omega & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \omega^2 \end{pmatrix} \begin{pmatrix} b_0 & b_1 & 0 \\ b_{-1} & 0 & b_1 \\ 0 & b_{-1} & -b_0 \end{pmatrix} \begin{pmatrix} \omega^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \omega \end{pmatrix} = \begin{pmatrix} b_0 & \omega b_1 & 0 \\ \omega^2 b_{-1} & 0 & \omega b_1 \\ 0 & \omega^2 b_{-1} & -b_0 \end{pmatrix}.$$

The grading subspaces corresponding to the 0th, 1st, and 2nd power of ω respectively are generated by

$$L_0 = \{B_0\} \qquad L_1 = \{B_1\} \qquad L_{-1} = \{B_{-1}\}. \tag{2.9}$$

All three subspaces are of dimension one, hence the grading cannot be further refined. It is said that such a grading is fine.

For the refinement of (2.8) we use the group $\mathbb{Z}_2 \times \mathbb{Z}_2$ built from the grading groups of (2.7) and (2.8). Indeed, the automorphisms given by the matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

commute so that they can be used simultaneously for the grading. It is convenient to introduce the following generators labelled by two component subscripts:

$$C_{10} = B_0 \qquad C_{11} = B_1 + B_{-1} \qquad C_{01} = B_1 - B_{-1}. \tag{2.10}$$

The $\mathbb{Z}_2 \times \mathbb{Z}_2$-grading decomposes $o(2, 1)$ into the sum of three one-dimensional subspaces,

$$L_{10} = \{C_{10}\} \qquad L_{11} = \{C_{11}\} \qquad L_{01} = \{C_{01}\}. \tag{2.11}$$

During the commutation the subscripts are added componentwise mod 2.

The non-equivalence of the gradings (2.9) and (2.11) follows, for example, from the fact that that the grading (2.11) is generated by semisimple elements, while in the grading (2.9) two of the generators are nilpotent.

Every other grading of $o(2,1)$ by cyclic groups is equivalent under the action of the Lie group $o(2,1)$ to one of the gradings above.

3. The Lie algebra $e(2,1)$ and its gradings

The Lie algebra $e(2,1)$ is the semidirect sum of $o(2,1)$ and the three-dimensional Abelian ideal of translations in space and time. Its generic element can be faithfully represented by

$$\begin{pmatrix} b_0 & b_1 & 0 & x_1 \\ b_{-1} & 0 & b_1 & x_0 \\ 0 & b_{-1} & -b_0 & x_{-1} \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.1}$$

The generators of $e(2,1)$ can be chosen as

$$B_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad B_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad B_{-1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$X_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad X_0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad X_{-1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Their commutation relations then are

$$\begin{array}{lll} [B_0, B_1] = B_1 & [B_0, B_{-1}] = -B_{-1} & [B_1, B_{-1}] = B_0 \\ [B_0, X_1] = X_1 & [B_0, X_0] = 0 & [B_0, X_{-1}] = -X_{-1} \\ [B_1, X_1] = 0 & [B_1, X_0] = X_1 & [B_1, X_{-1}] = X_0 \\ [B_{-1}, X_1] = X_0 & [B_{-1}, X_0] = X_{-1} & [B_{-1}, X_{-1}] = 0. \end{array}$$

The algebra $e(2,1)$ carries a natural $\mathbb{Z}_2$-grading which reflects its semidirect product structure. Such a grading is the eigenspace decomposition of the following transformation of order 2:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} b_0 & b_1 & 0 & x_1 \\ b_{-1} & 0 & b_1 & x_0 \\ 0 & b_{-1} & -b_0 & x_{-1} \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$
$$= \begin{pmatrix} b_0 & b_1 & 0 & -x_1 \\ b_{-1} & 0 & b_1 & -x_0 \\ 0 & b_{-1} & -b_0 & -x_{-1} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

so that we have the grading subspaces generated by

$$L_0 = \{B_0, B_1, B_{-1}\} \qquad L_1 = \{X_0, X_1, X_{-1}\} \tag{3.2}$$

belonging to the eigenvalues 1 and -1 respectively. We mention that in physical terms this grading is produced by the operators PT, i.e. parity times the time reversal.

Here L_0 is the subalgebra $o(2,1)$ and L_1 is its irreducible three-dimensional representation space. The linear transformations of L_0 act on L_1 by means of commutation (adjoint action). In particular,

$$I_1 = X_1 X_{-1} + X_{-1} X_1 - X_0^2$$

is the invariant form because

$$[L_0, X_1 X_{-1} + X_{-1} X_1 - X_0^2] = 0$$

where L_0 stands for any element of L_0. The other Casimir operator of $e(2,1)$ is

$$I_2 = B_0 X_0 + X_0 B_0 + B_1 X_{-1} + X_{-1} B_1 - B_{-1} X_1 - X_1 B_{-1}.$$

The automorphisms of $o(2,1)$ act also on the representation space L_1. The corresponding eigenspace decomposition is the simultaneous grading of $o(2,1)$ and of its representation space spanned by X_1, X_0, X_{-1}. Thus the $\mathbb{Z}_2$- and $\mathbb{Z}_3$-gradings of $o(2,1)$ are extended to $e(2,1)$. More precisely, one has

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} b_0 & b_1 & 0 & x_1 \\ b_{-1} & 0 & b_1 & x_0 \\ 0 & b_{-1} & -b_0 & x_{-1} \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} b_0 & -b_1 & 0 & x_1 \\ -b_{-1} & 0 & -b_1 & -x_0 \\ 0 & -b_{-1} & -b_0 & x_{-1} \\ 0 & 0 & 0 & 0 \end{pmatrix};$$

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} b_0 & b_+ + b_- & 0 & x+z \\ b_+ - b_- & 0 & b_+ + b_- & y \\ 0 & b_+ - b_- & -b_0 & x-z \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -b_0 & -b_+ + b_- & 0 & x-z \\ -b_+ - b_- & 0 & -b_+ + b_- & -y \\ 0 & -b_+ - b_- & b_0 & x+z \\ 0 & 0 & 0 & 0 \end{pmatrix};$$

$$\begin{pmatrix} \omega & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \omega^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} b_0 & b_1 & 0 & x_1 \\ b_{-1} & 0 & b_1 & x_0 \\ 0 & b_{-1} & -b_0 & x_{-1} \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \omega^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \omega & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} b_0 & \omega b_1 & 0 & \omega x_1 \\ \omega^2 b_{-1} & 0 & \omega b_1 & x_0 \\ 0 & \omega^2 b_{-1} & -b_0 & \omega^2 x_{-1} \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Here the three gradings decompose $e(2,1)$ into

$$\mathbb{Z}_2 : L_0 = \{B_0, X_1, X_{-1}\} \qquad L_1 = \{B_1, B_{-1}, X_0\}. \tag{3.3}$$

$$\mathbb{Z}_2 : L_0 = \{B_1 - B_{-1}, X_1 + X_{-1}\} \qquad L_1 = \{B_0, B_1 + B_{-1}, X_0, X_1 - X_{-1}\}. \tag{3.4}$$

$$\mathbb{Z}_3 : L_0 = \{B_0, X_0\} \qquad L_1 = \{B_1, X_1\} \qquad L_{-1} = \{B_{-1}, X_{-1}\}. \tag{3.5}$$

The gradings can be further refined using two gradings simultaneously. That is possible when two grading automorphisms commute. Thus we can combine any two $\mathbb{Z}_2$ gradings (3.2)–(3.4) into a $\mathbb{Z}_2 \times \mathbb{Z}_2$ grading. The grading subspaces are then labelled by two component subscripts, each read mod 2:

$$L_{00} = \{X_1 + X_{-1}\} \qquad L_{10} = \{B_1 - B_{-1}\}$$
$$L_{01} = \{B_0, X_1 - X_{-1}\} L_{11} = \{X_0, B_1 + B_{-1}\}. \tag{3.6}$$
$$L_{00} = \{B_0\} \qquad L_{10} = \{B_1, B_{-1}\} \qquad L_{01} = \{X_1, X_{-1}\} \qquad L_{11} = \{X_0\}. \tag{3.7}$$
$$L_{01} = \{B_0, X_0\} \qquad L_{11} = \{B_1 + B_{-1}, X_1 - X_{-1}\} \qquad L_{10} = \{B_1 - B_{-1}, X_1 + X_{-1}\}. \tag{3.8}$$

Similarly a $\mathbb{Z}_2 \times \mathbb{Z}_3$-grading is obtained by combining either (3.2) or (3.3) with (3.5):

$$\begin{array}{lll} L_{00} = \{B_0\} & L_{01} = \{X_1\} & L_{02} = \{X_{-1}\} \\ L_{10} = \{X_0\} & L_{11} = \{B_1\} & L_{12} = \{B_{-1}\}. \end{array} \tag{3.9}$$

Finally, one can combine all three $\mathbb{Z}_2$-gradings. The result is a fine $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$-grading. Its subspaces are labelled by three subscripts each read mod 2.

$$\begin{array}{lll} L_{010} = \{B_0\} & L_{011} = \{X_0\} & L_{110} = \{B_1 + B_{-1}\}, \\ L_{111} = \{X_1 - X_{-1}\} & L_{100} = \{B_1 - B_{-1}\} & L_{101} = \{X_1 + X_{-1}\}. \end{array} \tag{3.10}$$

It was pointed out in the introduction that the grading labels and the specific automorphisms, whose eigenvalues determine the label, are far from unique. Above we have an example of a $\mathbb{Z}_2 \times \mathbb{Z}_3$-grading. However, an identical decomposition of $e(2,1)$ is obtained as a $\mathbb{Z}_6$-grading in the following way, using $\kappa = \sqrt{\omega} = e^{2\pi i/6}$.

$$\begin{pmatrix} \kappa^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \kappa^4 & 0 \\ 0 & 0 & 0 & \kappa^3 \end{pmatrix} \begin{pmatrix} b_0 & b_1 & 0 & x_1 \\ b_{-1} & 0 & b_1 & x_0 \\ 0 & b_{-1} & -b_0 & x_{-1} \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \kappa^4 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \kappa^2 & 0 \\ 0 & 0 & 0 & \kappa^3 \end{pmatrix}$$
$$= \begin{pmatrix} b_0 & \kappa^2 b_1 & 0 & \kappa^5 x_1 \\ \kappa^4 b_{-1} & 0 & \kappa^2 b_1 & \kappa^3 x_0 \\ 0 & \kappa^4 b_{-1} & -b_0 & \kappa x_{-1} \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The eigenspaces are now labelled by sixth roots of one:

$$\begin{array}{lll} L_0 = \{B_0\} & L_1 = \{X_{-1}\} & L_2 = \{B_1\} \\ L_3 = \{X_0\} & L_4 = \{B_{-1}\} & L_5 = \{X_1\}. \end{array} \tag{3.11}$$

4. Graded contractions of $e(2,1)$: formulation of the problem

In previous sections we have found explicit grading decompositions of the Lie algebra $e(2,1)$. Two of them are not only fine, i.e. cannot be further refined, but also decompose the algebra into one-dimensional subspaces. Thus they define bases (up to a normalization) of the algebra. These bases are inherently preferred by the structure of the algebra. There is no other basis with the grading property which would not be equivalent to one of these. In the commutation of such basis elements, the result is a multiple of another basis element and not a linear combination of several of them.

Consider a general grading decomposition of a Lie algebra, $L_0 + L_1 + \cdots + L_{N-1}$, into the sum of eigenspaces of an automorphism of order N of the Lie algebra [21, 24]. Then symbolically we write

$$0 \neq [L_j, L_k] = L_{j+k} \qquad (j, k \bmod N). \tag{4.1}$$

The symbolics of the notation is in that it is a shorthand for a commutator of any element of L_j with any element of L_k, the result being an element of L_{j+k}. The inequality to zero means that not all commutators are zero. A deformation of the commutation relations, denoted by $[L_j, L_k]_\varepsilon$, is defined using the undeformed commutator and the parameters ε_{jk}:

$$[L_j, L_k]_\varepsilon := \varepsilon_{jk}[L_j, L_k] = \varepsilon_{jk} L_{j+k} \qquad (j, k, m \bmod N). \tag{4.2}$$

Clearly deformed commutators also have the grading property (4.1).

It was explained elsewhere [21, 24] that the result of such a deformation is a Lie algebra provided the deformation parameters satisfy the algebraic equations

$$\varepsilon_{jk} = \varepsilon_{kj} \qquad (j, k, m \bmod N) \tag{4.3}$$

$$\varepsilon_{jk}\varepsilon_{m,j+k} = \varepsilon_{km}\varepsilon_{j,k+m} = \varepsilon_{mj}\varepsilon_{k,j+m} \tag{4.4}$$

following respectively from the requirements of antisymmetry of the commutation operation and from the Jacobi identities.

A solution of our contraction problem is a set of parameters $\{\varepsilon_{jk}\}$ satisfying (4.3) and (4.4). It is convenient to visualize and to display the parameters as a matrix although we do not have much use for the conventional matrix operations in this context. There are always two trivial solutions which we disregard in most situations: (i) all $\varepsilon_{jk} = 1$ (the algebra remains the same); and (ii) all $\varepsilon_{jk} = 0$ (the resulting algebra is Abelian).

Two types of solutions of equations (4.3) and (4.4) can exist: continuous and discrete ones. Consider for example a relation obtained from (4.4) for $j = i$, $k = -i$, $m = 0$,

$$\varepsilon_{i-i}(\varepsilon_{0i} - \varepsilon_{0-i}) = 0.$$

If we have $\varepsilon_{0i} = \varepsilon_{0-i} = \lambda$, then λ (or ε_{i-i}) can vary continuously from its original value $\lambda = 1$ to $\lambda = 0$. However, if we have $\varepsilon_{0i} \neq \varepsilon_{0-i}$, then ε_{i-i} must be zero and cannot vary continuously to zero. As a matter of fact the 'original' value $\varepsilon_{i-i} = 1$ is not allowed in this case, so the term 'deformation' of a Lie algebra is more appropriate than 'contraction' for discrete solutions of (4.4).

In some cases we use more than one cyclic group to label the grading subspaces. Then the subscripts are multicomponent. Thus, for a $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$-grading, one should read $\varepsilon_{m,j+k}$ as $\varepsilon_{(m_1,m_2,m_3)(j_1+k_1,j_2+k_2,j_3+k_3)}$ where the brackets and commas are used to avoid ambiguity in separating the two labels and their components.

An all-important modification of the contraction equations (4.3), (4.4) arises when some commutators, say $[L_j, L_k]$, are identically zero before the deformation. Then ε_{jk} is not defined and all equations involving ε_{jk} have to be removed from (4.3), (4.4). Thus, for example, for both fine gradings of $e(2, 1)$, the 'diagonal' parameters ε_{jj} are not defined for any j.

Solutions of the contraction equations then provide the commutation relations of the deformed Lie algebra. Two solutions $\{\varepsilon_{jk}\}$ and $\{\varepsilon'_{jk}\}$ for a real Lie algebra are equivalent provided we have

$$\varepsilon'_{jk} = \frac{a_{j+k}}{a_j a_k}\varepsilon_{jk} \tag{4.5}$$

for all j, k, where a_j are some non-zero real numbers.

Solving the appropriate system of contraction equations and selecting among the results a set of non-equivalent solutions, is always a possible strategy for finding all contractions which preserve the chosen grading. However in some cases like the Lie algebra $e(2, 1)$, it is advantageous to exploit particular features of the algebra at hand, namely its structure of a semidirect sum. It is the way we have proceeded in this article. Let us now say more about it in general.

The idea is to split the bigger problem ($e(2, 1)$ contractions) into two smaller ones: (i) contractions of the subalgebra $o(2, 1)$ of 'homogeneous' transformations, and (ii) contractions of the action of $o(2,1)$ on its three-dimensional representation space which in this case is the 'inhomogeneous' part of $e(2,1)$, generated by X_1, X_0, X_{-1}.

A general method for simultaneous graded contractions of a Lie algebra and any of its representations was formulated in [23, 24]. Here we apply it to $e(2,1)$. Consider one of the gradings in which the Abelian subalgebra is split into separate grading subspaces, for example (3.8) or (3.10). The rows and columns of the corresponding grading parameter matrix (ε_{ij}) can be rearranged into a matrix of 3×3 blocks

$$\begin{pmatrix} E & P \\ P^T & 0 \end{pmatrix} \tag{4.6}$$

where $E = E^T$ contains the contraction parameters of the subalgebra $o(2,1)$, P consists of the contraction parameters between $o(2,1)$ and the translations, P^T is present because the overall matrix has to be symmetric (4.3) but otherwise contains the same parameters as P, and the presence of a 0 matrix reflects the Abelian nature of the translation subalgebra. The latter is the special feature of $e(2,1)$ which makes possible the two-stage solution of the contraction equations.

The first step is to consider only the subset of them containing parameters from E. That is really just the contraction of $o(2,1)$. With those parameters fixed (denoting them as before ε_{ij}), solve the rest of the equations containing parameters from P (denoting the parameters by ψ_{jk} in order to make the distinction). There is an important difference between parameters ε_{ij} and ψ_{jk}. In general $\psi_{jk} \neq \psi_{kj}$ and the symmetry of the matrix (4.6) is assured by the simultaneous presence of P and P^T.

5. Analysis of $\mathbb{Z}_3$ toroidal contractions

5.1. The Jacobi identities

In this section we investigate in detail the toroidal contractions of $e(2,1)$, corresponding to the $\mathbb{Z}_3$ toroidal grading of $o(2,1)$, together with the grading induced on the representation space of translations. In other words we decompose $e(2,1)$ as in (3.4), but distinguish the coefficients ε_{ik} and ψ_{ik} as in (4.6).

The standard physical basis of the Lie algebra $e(2,1)$ consists of the rotation generator L_3, the two Lorentz boosts K_1 and K_2, and the three space-time translations, P_1, P_2 and P_0. The three grading subspaces are chosen as:

$$\begin{aligned} 0 \quad &: K_1 = B_0 \qquad P_2 = X_0 \\ 1 \quad &: K_2 - L_3 = B_1 \qquad P_0 + P_1 = X_1 \\ -1 &: K_2 + L_3 = B_1 \qquad P_0 - P_1 = X_{-1}. \end{aligned} \tag{5.1}$$

The $e(2,1)$ commutation relations in this basis are given in the following table:

	B_0	B_1	B_{-1}	X_0	X_1	X_{-1}
B_0	0	B_1	$-B_{-1}$	0	X_1	$-X_{-1}$
B_1	$-B_1$	0	B_0	X_1	0	X_0
B_{-1}	B_{-1}	$-B_0$	0	X_{-1}	X_0	0

$[X_\mu, X_\nu] = 0.$ (5.2)

The deformed commutation relations are

$$[B_i, B_k]_\varepsilon = \varepsilon_{ik}[B_i, B_k] \tag{5.3}$$

$$[B_i, X_\alpha]_\varepsilon = \psi_{i\alpha}[B_i, X_\alpha]. \tag{5.4}$$

The matrix $\varepsilon \in \mathbb{R}^{3\times 3}$ is symmetric, $\varepsilon_{ik} = \varepsilon_{ki}$. The matrix $\psi_{i\alpha}$ has no definite symmetry properties.

The Jacobi identities must be satisfied by the deformed algebra (and are satisfied by the original one). This imposes bilinear relations on the contraction matrices ε and ψ, namely

$$\varepsilon_{i,k+\ell}\varepsilon_{k\ell} = \varepsilon_{k,i+\ell}\varepsilon_{i\ell} = \varepsilon_{\ell,i+k}\varepsilon_{ik} \tag{5.5}$$

$$\varepsilon_{ik}\psi_{i+k,\alpha} = \psi_{i,k+\alpha}\psi_{k\alpha} = \psi_{i\alpha}\psi_{k,i+\alpha}. \tag{5.6}$$

More specifically, relations (5.5) and (5.6) can be rewritten as

$$\varepsilon_{1-1}(\varepsilon_{01} - \varepsilon_{0-1}) = 0 \tag{5.7}$$

$$\psi_{1-1}(\varepsilon_{01} - \psi_{0-1}) = 0 \qquad \psi_{-11}(\varepsilon_{0-1} - \psi_{01}) = 0 \tag{5.8}$$

$$\psi_{10}(\varepsilon_{10} - \psi_{01}) = 0 \qquad \psi_{-10}(\varepsilon_{0-1} - \psi_{0-1}) = 0 \tag{5.9}$$

$$\varepsilon_{1-1}\psi_{01} - \psi_{-11}\psi_{10} = 0 \qquad \varepsilon_{1-1}\psi_{0-1} - \psi_{1-1}\psi_{-10} = 0 \tag{5.10}$$

$$\psi_{-11}\psi_{10} - \psi_{1-1}\psi_{-10} = 0. \tag{5.11}$$

5.2. Solution of the contraction equations

Our task now is to solve the relations (5.7)–(5.11) and then analyse all limits in which one or more of the constants $\varepsilon_{\mu\nu}$ or $\psi_{\mu\nu}$ are set equal to zero in a manner compatible with the above relations.

We start with relation (5.7). It has four classes of solutions. Indeed, the matrix

$$\varepsilon = \begin{pmatrix} \varepsilon_{00} & \varepsilon_{01} & \varepsilon_{0-1} \\ \varepsilon_{01} & \varepsilon_{11} & \varepsilon_{1-1} \\ \varepsilon_{0-1} & \varepsilon_{1-1} & \varepsilon_{-1-1} \end{pmatrix} \tag{5.12}$$

can have one of the following forms:

$$\dot{\varepsilon_1} = \begin{pmatrix} * & 1 & a \\ 1 & * & 0 \\ a & 0 & * \end{pmatrix} \qquad \varepsilon_2 = \begin{pmatrix} * & 0 & 0 \\ 0 & * & 1 \\ 0 & 1 & * \end{pmatrix} \tag{5.13}$$

$$\varepsilon_3 = \begin{pmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix} \qquad \varepsilon_4 = \begin{pmatrix} * & 1 & 1 \\ 1 & * & 1 \\ 1 & 1 & * \end{pmatrix}. \tag{5.14}$$

The matrices ε_1 and ε_2 correspond to non-trivial contractions of $0(2, 1)$, ε_3 and ε_4 to trivial ones. The asterisks in (5.13) and (5.14) are arbitrary numbers. The corresponding matrix elements multiply zeros in the commutation table.

Once relation (5.7) is solved, we can systematically solve the remaining ones, namely (5.8)–(5.11), either by hand, or in a computer assisted manner. For each matrix ε we shall present the possible matrices ψ, putting

$$\psi = \begin{pmatrix} * & b & c \\ \lambda & * & \mu \\ \rho & \sigma & * \end{pmatrix}. \tag{5.15}$$

From the commutation table (5.2) we see that the parameters b and c are eigenvalues of the element B_0. The entire first row in ε and ψ can be simultaneously scaled (by redefining B_0) but is otherwise fixed. The parameters λ, μ, ρ, σ, on the other hand, are either equal to zero, or can be independently scaled to the value 1 (again by redefining elements of the basis after contraction).

Let us run through the individual cases, listing only mutually inequivalent ones.

$\varepsilon = \varepsilon_1$: The constants a, b and c cannot be scaled. We list all possible forms of the matrix ψ, all unspecified constants are arbitrary:

$$\begin{array}{llllll}
\text{A1.} & \lambda = \mu = \rho = \sigma = 0 & & & \\
\text{A2}_1. & \lambda = \mu = \sigma = 0 & \rho = 1 & c = a & \\
\text{A2}_2. & \mu = \rho = \sigma = 0 & \lambda = 1 & b = 1 & \\
\text{A3}_1. & \lambda = \rho = \sigma = 0 & \mu = 1 & c = 1 & \\
\text{A3}_2. & \lambda = \mu = \rho = 0 & \sigma = 1 & b = a & \\
\text{A4.} & \lambda = \rho = 1 & \mu = \sigma = 0 & b = 1 & c = a \\
\text{A5}_1. & \lambda = \mu = 1 & \rho = \sigma = 0 & b = c = 1 & \\
\text{A5}_2. & \sigma = \rho = 1 & \lambda = \mu = 0 & b = c = a & \\
\text{A6.} & \mu = \sigma = 1 & \lambda = \rho = 0 & a = b & c = 1.
\end{array} \tag{5.16}$$

All algebras obtained by the above contractions will be solvable, but not nilpotent.

$\varepsilon = \varepsilon_2$: In this case relations (5.8)–(5.11) impose $b = c = 0$ in (5.15). Hence all contracted Lie algebras will be nilpotent.

The inequivalent possibilities for ψ are

$$\begin{array}{lll}
\text{A7.} & \lambda = \rho = 1 & \mu = \sigma = 0 \\
\text{A8.} & \lambda = \mu = 1 & \rho = \sigma = 0 \\
\text{A9.} & \mu = \sigma = 1 & \lambda = \rho = 0 \\
\text{A10.} & \lambda = 1 & \mu = \rho = \sigma = 0 \\
\text{A11.} & \lambda = \rho = 1 & \mu = \sigma = 0.
\end{array} \tag{5.17}$$

$\varepsilon = \varepsilon_3$: Although $0(2, 1)$ contracts to an Abelian algebra, its action on the translations can be non-trivial:

$$\begin{array}{lllll}
\text{A12.} & b = 1 & c \neq 0 & \lambda = \mu = \rho = \sigma = 0 & \\
\text{A13.} & b = 1 & c = 0 & \mu = 1 & \lambda = \rho = \sigma = 0 \\
\text{A14.} & b = 1 & c = 0 & \mu = \lambda = \rho = \sigma = 0 & \\
\text{A15.} & b = c = 0 & \lambda = \rho = 1 & \mu = \sigma = 0 & \\
\text{A16.} & b = c = 0 & \rho = \sigma = 1 & \lambda = \mu = 0 & \\
\text{A17.} & b = c = 0 & \mu = \sigma = 1 & \lambda = \rho = 0. & \\
\text{A18.} & b = c = 0 & \rho = 1 & \lambda = \mu = \sigma = 0. &
\end{array} \tag{5.18}$$

$\varepsilon = \varepsilon_4$: No contraction in the $0(2, 1)$ subalgebra

$$\text{A19.} \qquad b = c = 0 \qquad \lambda = \mu = \rho = \sigma = 0. \tag{5.19}$$

We have left out the two trivial contractions, obtained for $\varepsilon = \varepsilon_3$, $b = c = 0$, $\lambda = \mu = \rho = \sigma = 0$ and $\varepsilon = \varepsilon_4$, $b = c = 1$, $\lambda = \mu = \rho = \sigma = 1$, respectively. The first corresponds to the contraction $e(2, 1) \to$ Abelian, the second to $e(2, 1) \to e(2, 1)$.

5.3. Basis-independent identification of Lie algebras

The $\mathbb{Z}_3$ graded contractions of $e(2, 1)$ lead to a large variety of Lie algebras that are mutually non-isomorphic. We shall give an overview below, but first we list some classification criteria [26].

A Lie algebra L is decomposable if it can be written as a direct sum of two (or more) subalgebras

$$L = L_1 \oplus L_2 \qquad [L_1, L_1] \subseteq L_1 \qquad [L_2, L_2] \subseteq L_2 \qquad [L_1, L_2] = 0. \tag{5.20}$$

We shall present decomposable algebras in an already decomposed form, such that each component is indecomposable. Further we deal only with indecomposable Lie algebras.

The indecomposable Lie algebras that we obtain are either solvable, or nilpotent (we shall use the word ‘solvable’ to mean solvable, but not nilpotent).

A solvable Lie algebra L has a uniquely defined nilradical $NR(L)$, i.e. a maximal nilpotent ideal [17]. For each indecomposable solvable, or nilpotent Lie algebra we shall list the dimensions of Lie algebras in three series.

- The *derived series* (DS)

$$L^0 = L \qquad L_1^1 = [L^0, L^0] \qquad L^k = [L^{k-1}, L^{k-1}] \tag{5.21}$$

(the derived series terminates for some k ($L^k = 0$) for solvable and nilpotent Lie algebras).

- The *lower central series* (CS)

$$L^{(0)} = L \qquad L^{(1)} = [L^{(0)}, L^{(0)}], \ldots, L^{(k)} = [L^0, L^{k-1}]. \tag{5.22}$$

The lower central series terminates ($L^{(k)} = 0$) for some k for nilpotent Lie algebras, but not for other solvable ones.

The *upper central series* (US). We denote the centre of the Lie algebra $C(L)$ and then introduce a series of ‘higher centres’ of L, namely

$$C(L) \subset C^{(2)}(L) = C(L/C(L)) \subset C^{(3)} = C(L/C^{(2)}(L)) \subset \ldots. \tag{5.23}$$

This series terminates for nilpotent Lie algebras with $C^{(k)}(L) = L$ for some k.

5.4. Identification of the contracted Lie algebras

Class I. Solvable indecomposable (SI) (non-nilpotent) Lie algebras. These all come from contractions characterized by $\varepsilon = \varepsilon_1$ (see equations (5.13) and (5.16)). Their nilradicals are all five-dimensional. (Below, letters in brackets indicate real free parameters.)

SI1(a). Obtained from (A.4). The nilradical is indecomposable and we have

$$\begin{array}{llll} a \neq 0 & \text{DS}: (6, 4, 0) & \text{CS}: (6, 4, 4) & \text{US}: (0) \\ a = 0 & \text{DS}: (6, 3, 0) & \text{CS}: (6, 3, 2, 2) & \text{US}: (1, 2, 2). \end{array}$$

SI2(a). Obtained from (A.6). The nilradical is indecomposable, but not isomorphic to NR(SI1). We have

$$\begin{array}{llll} a \neq 0 & \text{DS}: (6, 5, 1, 0) & \text{CS}: (6, 5, 5) & \text{US}: (1) \\ a = 0 & \text{DS}: (6, 3, 1, 0) & \text{CS}: (6, 3, 3) & \text{US}: (1, 2, 2). \end{array}$$

SI3(a). Obtained from (A.5_1) and (A.5_2) for $a \neq 0$. The nilradical is decomposable, $NR(L) = \{B_1, X_0, X_1, X_{-1}\} \oplus \{B_{-1}\}$. We have

$$\text{DS} = (6, 5, 2, 0) \qquad \text{CS}: (6, 5, 5) \qquad \text{US}: (0).$$

SI4(a, b). Obtained from (A.2_1) with $c = a \neq 0, b \neq 0$; also from (A.2_2) with $c \neq 0, a \neq 0, c \to b$. The nilradical is decomposable, $NR(L) = \{X_{-1}, B_{-1}, X_0\} \oplus \{B_1\} \oplus \{X_1\}$. We have

$$\text{DS} = (6, 4, 0) \qquad \text{CS} = (6, 4, 4) \qquad \text{US} = (0).$$

SI5(a, b). Obtained from (A.3_1) with $a \neq 0$, $b \neq 0$; also from (A.3_2) with $a \neq 0$, $c \neq 0$, $c \to b$. The nilradical is decomposable, $NR(L) = \{X_0, B_1, X_{-1}) \oplus \{X_1\} \oplus \{B_{-1}\}$. We have

$$\text{DS} = (6, 5, 1, 0) \qquad \text{CS} = (6, 5, 5) \qquad \text{US} = (1, 1).$$

We see that the above Lie algebras are all mutually non-isomorphic. Only SI4 and SI5 have isomorphic nilradicals. However, their derived and central series involve different dimensions.

Class II. Nilpotent indecomposable (NI) Lie algebras. They all come from contractions associated with ε_2 (see (5.13) and (5.17)).

NI1. Obtained from (A.7). We have

$$\text{DS} = (6, 3, 0) \qquad \text{CS} = (6, 3, 0) \qquad \text{US} = (3, 6).$$

NI2. Obtained from (A.8). We have

$$\text{DS} = (6, 3, 0) \qquad \text{CS} = (6, 3, 1, 0) \qquad \text{US} = (2, 4, 6).$$

The Lie algebra has a five-dimensional Abelian ideal.

NI3. Obtained from (A.9). We have

$$\text{DS} = (6, 2, 0) \qquad \text{CS} = (6, 2, 0) \qquad \text{US} = (2, 6).$$

Class III. Solvable (non-nilpotent), decomposable (SD). They come from contractions associated with ε_1, or ε_3. We shall order them according to decomposition pattern.

- Decomposition $6 = 5 + 1$.

SD1(a, b, c). Obtained from (A.1) with $abc \neq 0$. The algebra is $\{B_0, B_1, B_{-1}, X_1, X_{-1}\} \oplus \{X_0\}$. The nilradical $\{B_1, B_{-1}, X_1, X_{-1}\}$ of the five-dimensional solvable Lie algebra is Abelian.

SD2(a). Obtained from (A.2_1) for $b = 0$, $c = a \neq 0$ and (A.2_2) for, $a = 0$, $c \neq 0$ or $c = 0$, $a \neq 0$. The nilradical of the five-dimensional solvable Lie algebra is decomposable according to the pattern $4 = 3 + 1$.

SD3(b). Obtained from (A.3_1) with $a = 0$, $b \neq 0$ or $a \neq 0$, $b = 0$. The nilradical of the five-dimensional Lie algebra is the same as for SD2(a). The algebras SD2 and SD3 are however not isomorphic. An isomorphic Lie algebra is obtained from (A3_2) with $a \neq 0$, $c = 0$.

SD4. Obtained from (A.5_1) with $a = 0$. The nilradical of the five-dimensional solvable Lie algebra $\{B_0, B_1, X_1, X_{-1}, X_0\}$ is indecomposable.

- Decomposition $6 = 4 + 2$.

SD5. Obtained from (A.5_2) with $a = b = c = 0$. The algebra $\{B_0, B_1\}$ is solvable, the four-dimensional algebra $\{B_{-1}, X_1, X_0, X_{-1}\}$ is nilpotent.

- Decomposition $6 = 4 + 1 + 1$.

SD6(a, b). Obtained from (A.1) when one of the parameters a, b, c is zero, the other two are not, e.g. $\{B_0, B_1, B_{-1}, X_1\} \oplus \{X_0\} \oplus \{X_{-1}\}$. The nilradical of the four-dimensional solvable Lie algebra is Abelian.

SD7. Obtained from (A.2_2) with $a = c = 0$. The nilradical of the four-dimensional solvable Lie algebra is non-abelian (the Heisenberg algebra). An isomorphic Lie algebra is obtained from (A.3_1) with $a = b = 0$.

- Decomposition $6 = 3 + 3$.

SD8(b). Obtained from (A2.$_1$) for $a = 0$. The algebra $\{B_0, B_1, X_1\}$ is solvable, $\{B_{-1}, X_{-1}, X_0\}$ is nilpotent. An isomorphic algebra is obtained from (A3.$_2$) with $a = 0$, $c \neq 0$.

- Decomposition $6 = 3 + 2 + 1$.

SD9. Obtained from (A2.$_1$) with $a = 0$, $b = 0$. We have $\{B_1, X_{-1}, X_0\} \oplus \{B_0, B_1\} \oplus \{X_1\}\}$. The first algebra is nilpotent, the second solvable. An isomorphic algebra is obtained from (A3.$_2$) with $a = c = 0$ and from (A13).

- Decomposition $6 = 3 + 1 + 1 + 1$.

SD10(a). Obtained from (A1) with two of the components a, b, c equal to zero, one not. An isomorphic algebra is obtained from (A12).

- Decomposition $6 = 2 + 1 + 1 + 1 + 1$.

SD11. Obtained from (A1) with $a = b = c = 0$ and also from (A14).

Class IV. Nilpotent, decomposable.
Such algebras can correspond to $\varepsilon = \varepsilon_2$ or $\varepsilon = \varepsilon_3$.

- Decomposition $6 = 5 + 1$.

ND1. From (A.10) we get

$$\{B_0, B_1, B_{-1}, X_0, X_1\} \oplus \{X_{-1}\}.$$

The invariant series for the first algebra are

$$\mathrm{DS}(5, 2, 0) \qquad \mathrm{CS} = (5, 2, 0) \qquad \mathrm{US} = (2, 5).$$

An isomorphic Lie algebra arises in the case A.15.

ND2. From (A.17) we get

$$\{B_1, B_{-1}, X_1, X_{-1}, X_0\} \oplus \{B_0\}$$

with

$$\mathrm{DS} = (5, 1, 0) \qquad \mathrm{CS} = (5, 1, 0) \qquad \mathrm{US} = (1, 5).$$

- Decomposition $6 = 4 + 1 + 1$.

ND3. From (A.16) we have

$$\{B_1, X_0, X_1, X_{-1}\} + \{B_0\} + \{B_{-1}\}$$

with

$$\mathrm{DS} = (4, 2, 0) \qquad \mathrm{CS} = (4, 2, 1, 0) \qquad \mathrm{CS} = \{1, 2, 4\}.$$

- Decomposition $6 = 3 + 1 + 1 + 1$.

ND4. From (A.11) we have

$$\{B_0, B_1, B_{-1}\} \oplus \{X_0\} \oplus X_1 \oplus X_{-1}\}.$$

An isomorphic Lie algebra is obtained from (A.18).

Class V. Not solvable, decomposable.

NS1. $\varepsilon = \varepsilon_4$, case (A.19) yields

$$\{B_0, B_1, B_{-1}\} + \{X_0\} \oplus \{X_1\} \oplus \{X_{-1}\}.$$

In this case $\{B_0, B_1, B_{-1}\}$ is $\mathrm{sl}(2, \mathbb{R})$ (acting trivially on the translations).

We note that above we have included both discrete and continuous contractions. The continuous ones are those corresponding to

ε_1 with $a = 1 \qquad b = c = 1$
ε_2 with $b = c = 0$
ε_3 with $b = c = 0$.

6. Contractions corresponding to the non-toroidal fine grading

6.1. Basis of the Lie algebra and the Jacobi identities

Let us consider the $\mathbb{Z}_2 \times \mathbb{Z}_2$ grading (3.9) of $e(2,1)$ and again follow the method indicated in equation (4.6). To simplify notations we shall denote the grading spaces as follows

$$A_1 \equiv L_{01} \sim \{K_1, P_2\} \qquad A_2 \equiv L_{11} \sim \{K_2, P_1\} \qquad A_3 \equiv L_{10} \sim \{L_3, P_0\}. \tag{6.1}$$

The grading is then cyclic

$$[A_1, A_2] \subseteq A_3 \qquad \text{cyclic.} \tag{6.2}$$

The $e(2,1)$ commutation relations in this 'physical' basis are

(6.3)

	K_1	K_2	L_3	P_2	P_1	P_0
K_1	0	$-L_3$	$-K_2$	0	P_0	P_1
K_2	L_3	0	K_1	P_0	0	P_2
L_3	K_2	$-K_1$	0	$-P_1$	P_2	0

The deformed commutation relations are given by two matrices

$$\varepsilon = \begin{pmatrix} * & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{12} & * & \varepsilon_{23} \\ \varepsilon_{13} & \varepsilon_{23} & * \end{pmatrix} \qquad \psi = \begin{pmatrix} * & \psi_{12} & \psi_{13} \\ \psi_{21} & * & \psi_{23} \\ \psi_{31} & \psi_{32} & * \end{pmatrix}. \tag{6.4}$$

The Jacobi identity for $\{K_1, K_2, L_3\}$ imposes no constraints on the matrix ε. The Jacobi identities involving two elements of $o(2,1)$ and one translation P_μ imply

$$\varepsilon_{13}\psi_{23} - \psi_{32}\psi_{13} = 0 \qquad \varepsilon_{13}\psi_{21} - \psi_{12}\psi_{31} = 0 \tag{6.5}$$

$$\varepsilon_{23}\psi_{13} - \psi_{31}\psi_{23} = 0 \qquad \varepsilon_{23}\psi_{12} - \psi_{21}\psi_{32} = 0 \tag{6.6}$$

$$\varepsilon_{12}\psi_{32} - \psi_{23}\psi_{12} = 0 \qquad \varepsilon_{12}\psi_{31} - \psi_{13}\psi_{21} = 0. \tag{6.7}$$

6.2. Relation between Inönü–Wigner and graded contractions of o(2,1)

The $0(2,1)$ contraction matrix ε is an arbitrary symmetric matrix. Hence its introduction provides an embedding of the algebra $0(2,1)$ into a continuous family of three-dimensional Lie algebras. Let us consider the different possibilities that occur.

(1) $\varepsilon_{12}\varepsilon_{23}\varepsilon_{13} \neq 0$.
By recalling the basis elements K_1, K_2 and L_3 we can transform ε in this case into e.g.

$$\varepsilon = \begin{pmatrix} * & \kappa_1 & 1 \\ \kappa_1 & * & \kappa_0 \\ 1 & \kappa_0 & 0 \end{pmatrix} \qquad \kappa_1 = \pm 1 \qquad \kappa_0 = \pm 1$$
$$\kappa_0 = \text{sign}(\varepsilon_{23}\varepsilon_{13}) \qquad \kappa_1 = \text{sign}\, \varepsilon_{12}\varepsilon_{13}. \tag{6.8}$$

The corresponding 'deformed' Lie algebra is $o(3)$ for $\kappa_1 = -1$, $\kappa_0 = 1$ and $o(2,1)$ for any other choice of signs. An Inönü–Wigner contraction would in this case simply be a change of basis in $o(2,1)$ and would imply $\varepsilon_{23}\varepsilon_{13} > 0$, $\varepsilon_{12}\varepsilon_{13} > 0$ (i.e. $\kappa_0 = \kappa_1 = 1$).

(2) One matrix element ε_{ik} vanishes.

(2a) $\varepsilon_{12} = 0$, $\varepsilon_{13} = 1$, $\varepsilon_{23} = \pm 1$.
The values of ε_{13} and ε_{23} are a result of a recalling. The two signs of ε_{23} correspond to the contractions

$$o(2,1) \to e(2) \qquad \varepsilon_{23} = 1$$
$$o(2,1) \to e(1,1) \qquad \varepsilon_{23} = -1 \tag{6.9}$$

where $e(2)$ and $e(1,1)$ are the Euclidean and pseudo-Euclidean Lie algebras in two dimensions, respectively.

(2b) $\varepsilon_{13} = 0$, $\varepsilon_{12} = 1$, $\varepsilon_{23} = \pm 1$.

In this case we have

$$\begin{aligned} &o(2,1) \to e(1,1) \qquad \varepsilon_{23} = 1 \\ &o(2,1) \to e(2) \qquad \varepsilon_{23} = -1. \end{aligned} \tag{6.10}$$

Inönü–Wigner contractions would lead to $\varepsilon_{23} = 1$ in both cases above.

(3) Two of the matrix elements ε_{ik} vanish.

In this case all three possible choices of two vanishing elements are equivalent. Moreover, the graded contractions are equivalent to the Inönü–Wigner ones. The contraction leads to the Heisenberg algebra, i.e. we have

$$[K_1, K_2]_\varepsilon = L_3 \qquad [L_3, K_1]_\varepsilon = [L_3, K_2]_\varepsilon = 0. \tag{6.11}$$

(4) All three ε_{ik} vanish.

The obtained algebra is Abelian.

6.3. Solution of the contraction equations

Let us now solve equations (6.5)–(6.7) for the matrix elements of the matrix ψ in equation (6.4). We shall first specify the choice of the matrix ε, as discussed in section 6.2. To simplify notations, we relabel the matrix ψ as

$$\psi = \begin{pmatrix} * & \lambda & \mu \\ \nu & * & \rho \\ \sigma & \tau & * \end{pmatrix}. \tag{6.12}$$

Below we give all inequivalent solutions for ε and ψ, already simplified by recalling and reordering basis elements of the contracted Lie algebra, whenever necessary.

(1)

$$\varepsilon = \begin{pmatrix} * & 0 & 1 \\ 0 & * & \kappa_0 \\ 1 & \kappa_0 & * \end{pmatrix} \qquad \kappa_0 = \pm 1 \tag{6.13}$$

A1. $\lambda = \nu = 0$, $\mu = \sigma = 1$, $\rho = \tau = \kappa_0$

A2. $\mu = \rho = 0$, $\lambda = \nu = \sigma = 1$, $\tau = \kappa_0$

A3. $\lambda = \mu = \nu = \rho = 0$, $\sigma = -a$, $\tau = \kappa_1 a$, $a \neq 0$, $\varepsilon_1 = \pm 1$

A4. $\lambda = \mu = \nu = \rho = 0$, $\sigma = 1$, $\tau = 0$, (or $\sigma = 0$, $\tau = 1$)

A5. $\lambda = \mu = \nu = \rho = 0$, $\sigma = \tau = 0$.

(2)

$$\varepsilon = \begin{pmatrix} * & 1 & 0 \\ 1 & * & \kappa_0 \\ 0 & \kappa_0 & * \end{pmatrix} \qquad \kappa_0 = \pm 1 \tag{6.14}$$

B1. $\mu = \sigma = 0$, $\lambda = \nu = 1$, $\rho = \tau = \kappa_0$

B2. $\lambda = \tau = 0$, $\mu = \nu = \sigma = 1$, $\rho = \kappa_0$

B3. $\lambda = \mu = \sigma = \tau = 0$, $\nu = a$, $\rho = \kappa_1 a$, $a \neq 0$, $\kappa_1 = \pm 1$

B4. $\lambda = \mu = \sigma = \tau = 0$, $\nu = 1$, $\rho = 0$(or $\nu = 0$, $\rho = 1$)

B5. $\lambda = \mu = \sigma = \tau = 0$, $\nu = \rho = 0$.

(3)

$$\varepsilon = \begin{pmatrix} * & 0 & 0 \\ 0 & * & 1 \\ 0 & 1 & * \end{pmatrix} \tag{6.15}$$

C1. $\lambda = \mu = \nu = \sigma = 0, \rho = \tau = 1$

C2. $\lambda = \mu = \rho = \tau = 0, \nu = \sigma = 1$

C3. $\lambda = \mu = \nu = \rho = 0, \sigma = \varepsilon_0, \tau = 1, \varepsilon_0 = \pm 1$

or $\lambda = \mu = \sigma = \tau = 0, \nu = \varepsilon_0, \rho = 1, \varepsilon_0 = \pm 1$

C4. $\lambda = \mu = \nu = \sigma = 0, \rho = 1, \tau = 0$, (or $\rho = 0, \tau = 1$)

C5. $\lambda = \mu = \rho = \tau = 0, \nu = 1, \sigma = 0$, (or $\nu = 0, \sigma = 1$)

C6. $\mu = \rho = \sigma = 0, \lambda = \nu = \tau = 1$, (or $\lambda = \nu = \tau = 0, \mu = \rho = \sigma = 1$)

C7. All elements $\psi_{ik} = 0$.

(4)

$$\varepsilon = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{6.16}$$

D1. $\lambda = \mu = \nu = \sigma = 0, \rho = \tau = 1$, (or $\lambda = \nu = \rho = \tau = 0, \mu = \sigma = 1$
or $\mu = \rho = \sigma = \tau = 0, \lambda = \nu = 1$)

D2. $\nu = \rho = \sigma = \tau = 0, \lambda = 1, \mu = \varepsilon_0, \varepsilon_0$
$= \pm 1$(or any permutation of rows in ψ)

D3. $\lambda = \mu = \rho = \tau = 0, \nu = \sigma = 1$(or any permutation of columns in ψ)

D4. One element in ψ equal to 1, all others zero (e.g. $\lambda = 1, \mu = \nu = \rho = \sigma = \tau = 0$)

D5. All elements $\psi_{ik} = 0$.

(5)

$$\varepsilon = \begin{pmatrix} * & \kappa_1 & 1 \\ \kappa_1 & 0 & \kappa_0 \\ 1 & \kappa_0 & * \end{pmatrix} \tag{6.17}$$

E1. $\psi = \varepsilon$

E2. $\psi = 0$.

In each case equations (6.5)–(6.7) expressing the Jacobi identities have two 'trivial' solutions: $\psi = \varepsilon$ and $\psi = 0$. We have included them above.

6.4. Identification of the contracted Lie algebras

We shall follow the criteria outlined in section 5.3. The list given below is a continuation of that in section 5.4 using the same notations.

Class I. Solvable indecomposable SI (non-nilpotent) Lie algebras.

SI6(κ_0). Obtained from (A1) and (B1).

The nilradical is five-dimensional and indecomposable and we have

DS : (6, 4, 0) CS : (6, 4, 4, . . .) US : (0).

For $\kappa_0 = 1$ in the case (A1) this is the inhomogeneous Galilei algebra. Its usual physical realization by vector fields is

$$L_3 \to L = y\partial_x - x\partial_y \qquad K_1 \to B_1 = -t\partial_x \qquad K_2 \to B_2 = -t\partial_y$$
$$P_0 = \partial_t \qquad P_1 = \partial_x \qquad P_2 = \partial_y. \tag{6.18}$$

For $\kappa_0 = +1$ in the case (B1) we obtain an inhomogeneous pseudo-Galilei Lie algebra that can be realized e.g. as

$$K_2 \to K = y\partial_t + t\partial_y \qquad K_1 \to \Pi_1 = x\partial_t \qquad L_3 \to \Pi_2 = x\partial_y$$
$$P_0 = \partial_t \qquad P_1 = \partial_x \qquad P_2 = \partial_y. \tag{6.19}$$

The case (A1) with $\kappa_0 = -1$ correspond to (6.19), (B1) with $\kappa_0 = -1$ to (6.18).

Furthermore, algebra (6.19) is isomorphic to algebra SI1($a = 1$) obtained from the toroidal grading. Algebra (6.18) is obtained from the non-toroidal grading only.

SI7(ε_0). Obtained from (A2) and (B2). The nilradical is five-dimensional and indecomposable and the invariant series are

DS : (6, 5, 1, 0) CS : (6, 5, 5, . . .) US : (1).

For $\kappa_0 = -1$ this algebra is isomorphic to SI2. For $\kappa_0 = +1$ it is not obtained from a toroidal grading.

Class II. Nilpotent indecomposable (NI) Lie algebras.

NI4. Obtained from (C1). We have

DS : (6, 2, 0) CS : (6, 2, 0) US : (2, 6).

Isomorphic to NI3.

NI5. Obtained from (C2). We have

DS : (6, 3, 0) CS : (6, 3, 0) US : (3, 6).

Isomorphic to NI1.

NI6. Obtained from (C6). We have

DS : (6, 3, 0) CS : (6, 3, 1, 0) US : (2, 4, 6).

This algebra has only a four-dimensional Abelian ideal and is hence not isomorphic to NI2, even though all dimensions of their invariant series coincide.

Class III. Solvable (non-nilpotent) decomposable (SD)

- Decomposition $6 = 5 + 1$.

SD12(κ_0, κ_1). Obtained from (A3) and (B3). The algebra obtained from (A3) is $\{K_1, K_2, L_3, P_1, P_2\} \oplus P_0$. The five-dimensional algebra has a four-dimensional Abelian nilradical $\{K_1, K_2, P_1, P_2\}$. For $\kappa_0 = \kappa_1 = -1$ this is isomorphic to SD1($a = 1, b = 1, c = 1$). Otherwise it is new. The contracted algebra (B3) is obtained by interchanging $\{L_3, P_0\} \leftrightarrow \{K_2, P_1\}$, $\kappa_0 \leftrightarrow -\kappa_0$, $\kappa_1 \leftrightarrow -\kappa_1$.

The dimensions in the invariant series of the five-dimensional Lie algebra are

DS : (5, 4, 0) CS : (5, 4, 4, . . .) US : (0).

SD13(ε_0). Obtained from (A4) and (B4) with $\{L_3, P_0, \kappa_0\} \leftrightarrow \{K_2, P_1, -\kappa_0\}$). The five-dimensional Lie algebra $\{K_1, K_2, L_3, P_1, P_2\}$ (for (A4)) again has a four-dimensional Abelian nilradical $\{K_1, K_2, P_1, P_2\}$, but the invariant series correspond to

$$\mathrm{DS}: (5, 3, 0) \qquad \mathrm{CS}: (5, 3, 2, 2, \ldots) \qquad \mathrm{US}: (1, 2).$$

An isomorphic Lie algebra is obtained from (C3) with $\{K_1, K_2\} \leftrightarrow \{P_1, P_2\}$.

- Decomposition $6 = 3 + 1 + 1 + 1$.

SD14(κ_0). Obtained from (A5). We have the algebra $\{K_1, K_2, L_3\} \oplus \{P_0\} \oplus \{P_1\} \oplus \{P_2\}$ with

$$[L_3, K_1] = K_2 \qquad [L_3, K_2] = -\varepsilon_0 K_1 \qquad [K_1, K_2] = 0$$

so the three-dimensional Lie algebra is $e(2)$ for $\kappa_0 = 1$, $e(1, 1)$ for $\kappa_0 = -1$. The same Lie algebra is obtained from B_5, with $\{L_3, \kappa_0\} \leftrightarrow \{K_2, -\kappa_0\}$. The case (D2) leads to an isomorphic Lie algebra with $(K_1, K_2) \leftrightarrow (P_1, P_2)$, or $\{L_3, K_1\} \rightarrow \{P_0, P_2\}$.

Class IV. Nilpotent, decomposable (ND).

- Decomposition $6 = 5 + 1$.

ND5. The cases (C4), (C5) and (D3) all lead to isomorphic Lie algebras in which the five-dimensional nilpotent subalgebra has a five-dimensional Abelian ideal and the invariant series satisfy

$$\mathrm{DS}: (5, 2, 0) \qquad \mathrm{CS}: (5, 2, 0) \qquad \mathrm{US}: (2, 5).$$

- Decomposition $6 = 3 + 1 + 1 + 1$.

ND6. The cases are (C6) and (D4) and the obtained three-dimensional Lie algebra is the Heisenberg one.

- Decomposition $6 = 6 \times 1$.

The case D5 leads to the Abelian Lie algebra.

Class V. Nonsolvable Lie algebras (NS). These correspond to the cases (E1) and (E2).

NS3. The case (E1) for $\kappa_0 = \kappa_1 = 1$ is the trivial contraction $e(2, 1) \rightarrow e(2, 1)$. We have $e(2, 1) \rightarrow e(3)$ for $\kappa_0 = 1, \kappa_1 = -1$; $e(2, 1) \rightarrow e(2, 1)$ otherwise.

NS4. The case (E2) leads to the decomposition $e(2, 1) \rightarrow o(2, 1) + 3A_1$ for $(\kappa_0, \kappa_1) = (1, 1), (-1, -1), (-1, 1)$ and to $o(3) + 3A_1$ for $(\kappa_0, \kappa_1) = (1, -1)$.

7. Conclusions

The $\mathbb{Z}_3$ grading (3.5) used in section 5 is not a fine one: it can be further refined into the $\mathbb{Z}_2 \times \mathbb{Z}_3$ grading (3.5), or equivalently the $\mathbb{Z}_6$ grading (3.11). We use the method represented by eq. (4.6) which is equivalent to taking the fine grading. We have obtained 22 types of contracted Lie algebras, including five indecomposable solvable non-nilpotent ones, three indecomposable nilpotent ones, nine decomposable solvable non-nilpotent ones, four decomposable nilpotent ones and one that is not solvable (the direct sum of $o(2, 1)$ and a three-dimensional Abelian algebra).

Among these algebras, some are already obtained from a coarser grading, namely a $\mathbb{Z}_2$ one with

$$L_0 \sim \{B_0, X_0\} \qquad L_1 \sim \{B_1, B_{-1}, X_1, X_{-1}\}. \tag{7.1}$$

The corresponding contracted Lie algebras are obtained from those in our list by setting $a = 1$, $b = c$, $\lambda = \rho$ throughout. We obtain the solvable indecomposable algebras SI1($a = 1$), SI2, SI3($a = 1$), SI4($a = 1, b = 1$), SI5($a = 1, b = 1$). Similarly, the indecomposable nilpotent Lie algebras NI1 and NI3 (but not NI2 correspond to the grading (7.1). Among the decomposable ones the $\mathbb{Z}_2$ grading (7.1) provides SD1($a = 1, b = c \neq 0$), SD7, SD5($c = 0$), SD9($a = 1$), ND2, ND4, and NS1.

The matrix ε_1 in equation (5.13) corresponds to a continuous contraction for $a = 1$ only. In this case $o(2, 1)$ contracts to the Lie algebra of the pseudo-Euclidean group $P(1, 1)$ (i.e. B_1 and B_{-1} commute like translations, B_0 continues to act like a Lorentz boost). Equation (5.16) shows that the obtained solvable Lie algebra can act in many different ways on the space-time translations X_0, X_1, X_{-1}.

The matrix ε_2 in equation (5.13) corresponds to a continuous contraction of $o(2, 1)$ to a Heisenberg algebra. Again equation (5.17) shows the different possible actions of this Heisenberg algebra on the translations.

Equation (5.18) in turn shows that different ways in which the Abelian algebra, corresponding to ε_3 can act on the translations. On the other hand, $o(2, 1)$, corresponding to ε_4 (no contraction in the $o(2, 1)$ part) can only act irreducibly (as in $e(2, 1)$), or trivially, as in equation (5.19).

The $\mathbb{Z}_2 \times \mathbb{Z}_2$ non-toroidal grading (3.9) used in section 6 is again not a fine one, but our treatment is totally equivalent to using the fine grading (3.10).

The non-toroidal contractions are all continuous ones, in that the values of the parameters ε_{ik} and ψ_{ik} in equation (7.4) can be varied continuously, while satisfying equations (6.5)–(6.7). However these contractions are not necessarily generalized Inönü–Wigner ones, in that they cannot all be viewed as singular changes of basis (over the field of real numbers $\mathbb{R}$). To see this, consider the $o(2, 1)$ contraction matrix ε of equation (6.8). This represents a Lie algebra for any values of κ_0 and κ_1. For $\kappa_0 = 1$, $\kappa_1 = -1$ this algebra is $o(3)$ (also for any $\kappa_0 > 0$, $\kappa_1 < 0$), for all other cases satisfying $\kappa_0\kappa_1 \neq 0$ it is $o(2, 1)$. Changes of basis correspond to all cases $\kappa_0 > 0$, $\kappa_1 > 0$. E.g. $\kappa_0 = -1$, $\kappa_1 = 1$ also corresponds to $o(2, 1)$, but the $o(2)$ generator L_3 and $o(1, 1)$ generator K_2 are interchanged. This is clearly not a change of basis over $\mathbb{R}$ (it would be over $\mathbb{C}$).

For $\kappa_1 = 0$, $\kappa_0 = 1$ we obtain the algebra $e(2)$ and the contraction is an Inönü–Wigner one. On the other and $\kappa_1 = 0$, $\kappa_0 = -1$ leads to $e(1, 1)$ and this is not a (singular) change of basis. Vice versa, if we take $\kappa_0 = 0$, $\kappa_1 = 1$ we obtain $e(1, 1)$ as an Inönü–Wigner contraction, whereas $\kappa_0 = 0$, $\kappa_1 = -1$ yields $e(2)$ (not a change of basis).

In any case, the introduction of the contraction matrices ε and Ψ embeds $o(2, 1)$ into a large family of mutually non-isomorphic algebras.

We note that the contractions corresponding to the two different fine gradings in general give different Lie algebras, mutually isomorphic only in special cases.

Let us say a few words about physical applications.

(1) Probably the most important algebra obtained by graded contractions of $e(2, 1)$ is the inhomogeneous Galilei algebra (6.18). It is obtained from the non-toroidal grading, but not the toroidal one. It could have been obtained using a coarse non-toroidal grading, namely (a $\mathbb{Z}_2$ one) with

$$L_0 = (L_3, P_0) \qquad L_1 = \{K_1, K_2, P_1, P_2\} \tag{7.2}$$

$$\varepsilon = \begin{pmatrix} * & 1 \\ 1 & 0 \end{pmatrix} \qquad \psi = \begin{pmatrix} * & 1 \\ 1 & 0 \end{pmatrix}. \tag{7.3}$$

(2) The ‘pseudo-Galilei’ Lie algebra (6.19) is accessible from $e(2, 1)$ via toroidal and non-toroidal contractions. This is the Lie algebra SI($a = 1$) and also (6.19).

In both cases it already appears for coarser gradings, e.g.

$$L_0 = (K_2, P_1) \qquad L_1 = \{L_3 K_1, P_0, P_2\} \tag{7.4}$$

with ε and ψ as in equation (7.3).

(3) The Casimir operators, or more generally the invariants of the coadjoint representation of the contracted Lie algebras can be calculated directly. For any Lie algebra with commutation relations

$$[X_i, X_k] = c_{ik}^{\ell} X_\ell \qquad 1 \leqslant i, k, \ell \leqslant n \tag{7.5}$$

these invariants satisfy

$$X_i F(x_1, \ldots, x_n) = 0 \qquad i = 1, \ldots, n$$
$$X_i = c_{ik}^{\ell} x_\ell \frac{\partial}{\partial x_k}$$

where $\{x_1, \ldots x_N\}$ are some (commuting) variables. For Inönü–Wigner contractions we can also introduce the contraction constants into the invariants I_1 and I_2 of $e(2, 1)$ and then obtain the invariants of the contracted Lie algebras in the appropriate limits.

Here we just present the invariants of the Lie algebras SI1–SI7

$$\mathrm{SI1(a)} : I_1 = x_{-1} x_1^a \qquad I_2 = (b_1 x_{-1} - b_{-1} x_1) x_1^{a-1}.$$

For continuous contractions we have $a = 1$.

$$\mathrm{SI2} : I_1 = x_0 \qquad I_2 = b_0 x_0 + b_1 x_{-1} - b_{-1} x_1$$
$$\mathrm{SI3(a)} : I_1 = x_0^2 - 2 x_1 x_{-1} \qquad I_2 = b_{-1} x_1^a$$
$$\mathrm{SI4(a,b)} : I_1 = x_1 x_{-1} b_1^{c-b} \qquad I_2 = x_{-1} b_1^c$$
$$\mathrm{SI5(a,b)} : I_1 = x_0 \qquad I_2 = x_1^a b_{-1}^b \qquad ab \neq 0$$
$$\mathrm{SI6}(\kappa_0) : I_1 = p_1^2 + \kappa_0 p_2^2 \qquad I_2 = k_1 p_2 - k_2 p_1$$
$$\mathrm{SI7}(\kappa_0) : I_1 = p_0 \qquad I_2 = p_0 \ell_3 + p_1 k_2 - \kappa_0 k_1 p_2.$$

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